THE BRAID GROUP

PABLO FELIPE ARDILA ROJO

Braid group definitions.

Definition 1 (geometric)**.** *Take 2 parallel planes L and L^0 of $\mathbb{R}^3$, a set of fixed points $\{P_1, \cdots, P_n\}$ in the plane L, and a set of fixed points $\left\{P_1^0, \cdots, P_n^0\right\}$ in the plane L^0. A Geometric Braid is a set of n arcs $A = \{A_1, ..., A_n\}$ embedding in $\mathbb{R}^3$, where the arc connecting point P_i on the upper level L with the point $P_{i(\upsilon)}^0$ on the lower level L^0, for some permutation υ of S_n such that:*

1. Each arc A_i intersects each intermediate parallel planes L and L^0

2. The arcs $A_1, \cdots, A_n$ intercepts each intermediate parallel plane between the planes L and L^0 in exactly n points

Definition 2. *The n-Braid Algebriac Group is the has the presentation with generators $\sigma_1, ..., \sigma_{n-1}$ and the relations*

$$\begin{aligned}\sigma_i\sigma_j &= \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \text{ and}\\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} \text{ if } 1 \leq i \leq n-2\end{aligned}$$

Definition 3 (topological)**.** *Let X be a topological space, define the n-configuration space X as the set*

$$F_n(X) = F_n(X, n) = \{(x_1, \cdots, x_n) \in X \times \cdots \times X \mid x_i \neq x_j \text{ for } i \neq j\}$$

and the quotient space is defined as

$$C_n(X) = F_n(X)/S_n$$

The n-Braid Group is defined as the topological fundamental group $\pi_1(C_n(\mathbb{C}))$. The group $\pi_1(F_n(\mathbb{C})$ is called the Topological Group of Pure Braids.

Definition 4 (geometryc-algebraic)**.** *Let $P_n(z)$ be the set of all monic polynomials of degree n in $\mathbb{C}[z]$.*

Consider the space Σ_n of all polynomials with multiple roots and define the set $P_n(z)\diagdown\Sigma_n$, the Braid Group is $\pi_1(P_n(Z)\diagdown\Sigma_n)$.

The Following is the main result that can proof the equivalence between the above definitions.

Theorem 5 (Artin Theorem)**.** *The Geometrical Braid Group $B(n)$ admits a presentation with generators $\sigma_1, ..., \sigma_{n-1}$ and relations*

$$\begin{aligned}\sigma_i\sigma_j &= \sigma_j\sigma_i \text{ if } |i-j| \geq 2 \text{ and}\\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1}\end{aligned}$$

Braid group representations. In the study of the structure of a group the role of representations is central. This section will discuss some of them.

(1) Representation in the Symmetric Group S_n, defined as follows

$$\tau_n : B_n \to S_n$$
$$\sigma_i \to (i, i+1)$$

(2) Representation in the Group of Automorphisms of a free group $F(a_1, a_2, \cdots, a_n)$

$$\begin{array}{rcccc} \gamma_n & : & B_n & \rightarrow & Aut(F(a_1, a_2, \cdots, a_n)) \\ & & \sigma_j & \rightarrow & g_j \end{array}$$

where, for each $1 \leq j \leq n-1$, g_j is an automorphism defined by

$$\begin{array}{rcl} g_j & : & F(a_1, a_2, \cdots, a_n) \rightarrow F(a_1, a_2, \cdots, a_n) \\ a_k & \rightarrow & (a_k)g_j = \begin{cases} a_k & \text{if k} \neq j, j+1 \\ a_j a_{j+1} a_j^{-1} & \text{if } k = j \\ a_j & \text{if } k = j+1 \end{cases} \end{array}$$

(3) The Burau Representation

Definition 6. *The Burau Representation is the homomorphism $\theta : B_n \rightarrow Gl_n(Z[t, t^{-1}])$ such that*

$$(\sigma_r)\theta = I_{r-1} \oplus \left[\begin{array}{cc} 1-t & t \\ 1 & 0 \end{array} \right] \oplus I_{n-r-1}$$

where I_{r-1} is the identity matrix of order $(r-1) \times (r-1)$ and I_{n-r-1} is the identity matrix of order $n-r-1$, also has to

$$\begin{array}{rcl} \theta_r \theta_s & = & \theta s \theta_r \text{ if } |r-s| \geq 2 \\ \theta_r \theta_{r+1} \theta_r & = & \theta_{r+1} \theta_r \theta_{r+1} \end{array}$$

Note that if we take the column vector $v = \left[\begin{array}{c} 1 \\ 1 \\ \vdots \\ 1 \end{array} \right] \in (Z[t, t^{-1}])^n$, *we have that for all r, $1 \leq 1 \leq r \leq n-1$.*

$$(\sigma_r)v = v$$

so the submodule generated by v is invariant under the representation θ, so the Burau Representation is reducible.

Lemma 7. *Burau representation can be decomposed as the sum of 1-dimensional representation and an irreducible representation of dimension $(n-1)$, $\theta : B_n \rightarrow Gl_{n-1}(Z[t, t^{-1}])$ such that*

$$(\sigma_r)\theta = \left[\begin{array}{cccc} -t & 1 & & 0 \\ o & 1 & & 0 \\ & & \ddots & \\ 0 & 0 & & 1 \end{array} \right] \text{ and } (\sigma_r)\theta = \left[\begin{array}{ccccccc} 1 & & & & \cdots & & \\ & \ddots & & & & & \\ & & 1 & 0 & 0 & & \\ & & t & -t & 1 & & \\ & & 0 & 0 & 1 & & \\ & & & & & \ddots & \\ & & & & & & 1 \end{array} \right]$$

where $-t$ is at position (r, r). The Representation $\theta : B_n \rightarrow Gl_{n-1}(Z[t, t^{-1}])$ is called the reduced Burau representation.

Recall that a representation is faithful if it is a homomoirfismo injective. After having defined the Burau representation is still the question of whether the faithful, ie if the homomorphism is injective.This question amounts to ask whether the braid groups are linear.

Results

For $n = 3$ reduced Burau representation is faithful. In 1991 John Moody proves that the Burau Representation is not faithful for $n \geq 9$. A year later D. D Long and M. Paton refining the ideas of Moody showed that the Burau Representation is not faithful for $n \geq 6$. Bigelow then shown for the case $n = 5$.

For the case $n = 4$, the question remains unresolved, there is indeed an outcome that links this problem with the theory of knots, which states: The Jones Polynomial of one variable is 1 for a non trivial knot if and only if the Burau Representation is faithful for $n = 4$.

(4) Representation on the Hecke Algebras

Let $q \in C$, the Hecke algebra $H(q, n)$ type A_{n-1}, is defined as the C-algebra generated by $g_1, g_2, \cdots, g_{n-1}$ and the relations

$$\begin{aligned} g_i^2 &= (q-1)g_i + q \\ g_i g_{i+1} g_1 &= g_{i+1} g_i g_{i+1} \\ g_i g_j &= g_j g_i, |i - j| \geq 2 \end{aligned}$$

by the Von Dick theorem is induced homorphism ρ of B_n in $H(q, n)$

$$\begin{aligned} \rho \quad : \quad B_n &\rightarrow H(q, n) \\ \sigma_i &\rightarrow g_i \end{aligned}$$

preserving relations in B_n, and therefore we have a representation of B_n.

An important application of the Representation of Hecke is a powerful knot invariant called the Jones Polynomial.

E-mail address: pabloardila@itm.edu.co

Instituto Tecnológico Matropolitano De Medellín, Colombia