INTRODUCTION TO VERTEX ALGEBRAS

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Abstract. The aim of the course will be to give a basic introduction to the field of vertex algebras. Vertex algebras were introduced by Borcherds in 1986, in his study of the Monster group, but were largely used by physicists at the time as en- coding the chiral part of a quantum field theory known as the 2-dimensional sigma model. In the mathematics literature the theory of vertex algebras was largely developed to study representations of infinite dimensional Lie algebras, principally afine Kac-Moody Lie algebras and superconfomal algebras. Most of the examples (if not all) that we will give in this class will be vertex algebra structures on vacuum (or induced highest weight) modules of infinite dimensional Lie superalgebras. It is therefore natural that we will spend three lectures developing the tools and the language needed to make sense of the def- inition of vertex algebras. We will go over the basics of representation theory of finite simple Lie algebras and then loop algebras. After this we will move onto the definition of a vertex algebra, and prove that the vacuum module for an afine Kac-Moody Lie algebra possesses such a structure. We will finish this class by proving the classical statement of Boson-Fermion correspondence, which incidentally is responsible for the name "vertex operators".

Lecture 1: (affine) Lie algebras
- Lie algebra definition and classical finite dimensional examples.
- Triangular decompositions, Borel and Cartan subalgebras.
- Representations, induced modules, highest weight representations.
- Loop and Afine Kac-Moody Lie algebras, untwisted examples, their Vac uum modules. Definition of Level.
- Algebras of vector fields, Virasoro Lie algebra, central charge.
- Superalgebras $\mathbb{Z}/2\mathbb{Z}$-graded algebras.

Lecture 2: Formal distributions
- Formal Diracś delta function. Properties.
- Locality: derivatives of the delta function, commutators, OPE and j-th products ($j \geq 0$) of local distributions.
- Formal Fourier transform and λ-brackets.
- Formal distribution Lie superalgebras.

Lecture 3: Conformal Lie algebras
- Definition, relation to formal distribution Lie superalgebras.
- Examples: Affine algebras, Heisenberg (free Bosons), free Fermions, Virasoro, Neveu-Schwarz.
- Annihilation subalgebras, vacuum modules.
- Normally ordered products, a prelude to vertex algebras.
- Non-commutative Wick formula as a Leibniz rule.
- Conformal embeddings.

Lecture 4: Vertex algebras
- Quasicommutativity, Quasiassociativity.
- Vertex Algebra definition: State-Field correspondence.
- Vertex Algebra definition as quantum diferential Poisson algebra.
- Freely generated examples.
- Lattice vertex algebras.
- Boston-Fermion correspondence.

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